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P_k -Factorization of Complete Bipartite Graphs

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1. Introduction

Let P_k be a path on k points and $K_{m,n}$ be a complete bipartite graph with partite sets V_1 and V_2 , where $|V_1|=m$ and $|V_2|=n$. A spanning subgraph F of $K_{m,n}$ is called a P_k -factor if each component of F is isomorphic to P_k . If $K_{m,n}$ is expressed as a line-disjoint sum of P_k -factors, then this sum is called a P_k -factorization of $K_{m,n}$.

In this paper, a necessary condition for the existence of a P_k -factorization of $K_{m,n}$ will be given. And it will be shown that the necessary condition is also sufficient when k is even.

2. P_k -Factor of $K_{m,n}$

With respect to a P_k -factor of $K_{m,n}$, we give the following theorem.

Theorem 1. A $K_{m,n}$ has a P_k -factor if and only if

(I) $m = n \equiv 0 \pmod{k/2}$ when k is even, and

(II) $m + n \equiv 0 \pmod{k}$, $(k-1)m \leq (k+1)n$ and $(k-1)n \leq (k+1)m$ when k is odd.

Proof. (Necessity) Suppose that $K_{m,n}$ has a P_k -factor F .

Let t be the number of components of F . Then $t = (m+n)/k$. Each

component is a path obtained by traversing V_1 and V_2 . Thus when k is even, it holds that $m=n=kt/2$. Condition (I) is necessary. And when k is odd, let t_1 (t_2) be the number of components of F whose end points are in V_1 (V_2), respectively. Then it holds that $m=((k+1)t_1+(k-1)t_2)/2$ and $n=((k-1)t_1+(k+1)t_2)/2$. So we have $t_1=((k+1)m-(k-1)n)/2k$ and $t_2=((k+1)n-(k-1)m)/2k$. From $0 \leq t_1 \leq t$ and $0 \leq t_2 \leq t$, we must have $(k-1)m \leq (k+1)n$ and $(k-1)n \leq (k+1)m$. Condition (II) is necessary.

(Sufficiency) When k is even, put $m=n=kt/2$. Consider a Hamilton-path of $K_{n,n}$ and divide it into t paths of same length. Then they form a P_k -factor of $K_{n,n}$. When k is odd, for those parameters m and n satisfying (II), put $t_1=((k+1)m-(k-1)n)/2k$ and $t_2=((k+1)n-(k-1)m)/2k$ and $t=(m+n)/k$. Then t_1 and t_2 are integers such as $0 \leq t_1 \leq t$ and $0 \leq t_2 \leq t$. And it holds that $m=((k+1)t_1+(k-1)t_2)/2$ and $n=((k-1)t_1+(k+1)t_2)/2$. Using $(k+1)t_1/2$ points in V_1 and $(k-1)t_1/2$ points in V_2 , consider t_1 P_k 's whose end points are in V_1 . Using remaining $(k-1)t_2/2$ points in V_1 and remaining $(k+1)t_2/2$ points in V_2 , consider t_2 P_k 's whose end points are in V_2 . Then these t_1+t_2 P_k 's are line-disjoint and they form a P_k -factor of $K_{m,n}$.

Corollary 1. A $K_{n,n}$ has a P_k -factor if and only if

(I)' $n \equiv 0 \pmod{k/2}$ when k is even, and

(II)' $n \equiv 0 \pmod{k}$ when k is odd.

3. P_k -Factorization of $K_{m,n}$

With respect to a P_k -factorization of $K_{m,n}$, we give the following theorem.

Theorem 2. If $K_{m,n}$ has a P_k -factorization, then it holds that

(I)" $m = n \equiv 0 \pmod{k(k-1)/2}$ when k is even, and

(II)" $m + n \equiv 0 \pmod{k}$, $(k-1)m \leq (k+1)n$, $(k-1)n \leq (k+1)m$
and $kmn / (k-1)(m+n)$ is an integer when k is odd.

Proof. Suppose that $K_{m,n}$ has a P_k -factorization. Let r be the number of P_k -factors of $K_{m,n}$ and t be the number of components of each P_k -factor. Then $t = (m+n)/k$ and $r = kmn / ((k-1)(m+n))$. Thus t and r are integers. By Theorem 1, it holds that $m = n \equiv 0 \pmod{k(k-1)/2}$ when k is even, and that $m + n \equiv 0 \pmod{k}$, $(k-1)m \leq (k+1)n$, $(k-1)n \leq (k+1)m$ and $kmn / ((k-1)(m+n))$ is an integer when k is odd.

Corollary 2. If $K_{n,n}$ has a P_k -factorization, then it holds that

(I)'" $n \equiv 0 \pmod{k(k-1)/2}$ when k is even, and

(II)'" $n \equiv 0 \pmod{2k(k-1)}$ when k is odd.

We prepare the following extension theorem, which is very useful.

Theorem 3. If $K_{m,n}$ has a P_k -factorization, then $K_{sm,sn}$ has a P_k -factorization for every positive integer s .

Proof. If every subgraph $K_{1,1}$ of $K_{s,s}$ is replaced by $K_{m,n}$, then $K_{s,s}$ is replaced by $K_{sm,sn}$. Using $K_{1,1}$ -factorization (1-factorization) of $K_{s,s}$, we can see that $K_{sm,sn}$ has a $K_{m,n}$ -factorization. Using a P_k -factorization of $K_{m,n}$, we can easily construct a P_k -factorization of $K_{sm,sn}$. About a 1-factorization of $K_{s,s}$, see [1,2].

Using this theorem, we can obtain several results. When k is even, we have the following lemma.

Lemma 1. k is even and $m = n = k(k-1)/2$

$\implies K_{m,n}$ has a P_k -factorization.

Proof. The proof is shown by a construction algorithm. Let

$V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}\}$, where $m = n = k(k-1)/2$. Construct $k-1$ P_k 's such as $P_k^{(i)} = v_{(i-1)a+1}^{(1)} v_{(i-1)b+1}^{(2)} v_{(i-1)a+2}^{(1)} v_{(i-1)b+2}^{(2)} \dots v_{ia-1}^{(1)} v_{ib}^{(2)} v_{ia}^{(1)} v_{k(i)}^{(2)}$, where $a = k/2$, $b = k/2 - 1$ and $k(i) = ((k/2 - 1) + 1 \bmod k - 1) + (k/2 - 1)(k - 1)$. Then $F = P_k^{(1)} \cup P_k^{(2)} \cup \dots \cup P_k^{(k-1)}$ is a P_k -factor. Increasing all point numbers of F in V_1 by $k-1 \pmod{m}$ simultaneously $k/2$ times and increasing all point numbers of F in V_2 by $k-1 \pmod{n}$ simultaneously $k/2$ times, we obtain $k^2/4$ P_k -factors. Then it can be easily checked that these P_k -factors are line-disjoint and that the sum of them is a P_k -factorization of $K_{m,n}$.

Applying Theorem 3 to Lemma 1 and considering Theorem 2, we have the following theorem.

Theorem 4. When k is even, a $K_{m,n}$ has a P_k -factorization if and only if $m = n \equiv 0 \pmod{k(k-1)/2}$.

When k is odd, we have the following lemmas.

Lemma 2. k is odd, $(k-1)m = (k+1)n$ and $kmn / (k-1)(m+n)$ is an integer

- \Rightarrow (i) $m + n \equiv 0 \pmod{k}$, and
(ii) $m = (k+1)s/2$, $n = (k-1)s/2$ when $k \equiv 3 \pmod{4}$,
 $m = (k+1)s$, $n = (k-1)s$ when $k \equiv 1 \pmod{4}$,
where s is a positive integer.

Lemma 3. k is odd, $(k-1)n = (k+1)m$ and $kmn / (k-1)(m+n)$ is an integer

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 $m = (k-1)s$, $n = (k+1)s$ when $k \equiv 1 \pmod{4}$,
where s is a positive integer.

Lemma 2 and Lemma 3 can be easily checked. We have the following

lemmas.

Lemma 4. $k \equiv 3 \pmod{4}$, $m = (k-1)/2$, $n = (k+1)/2$

$\Rightarrow K_{m,n}$ has a P_k -factorization.

Proof. The proof is shown by a simple construction algorithm.

Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}\}$, where $m = (k-1)/2$ and $n = (k+1)/2$. Construct a P_k such as $P_k = v_1^{(2)} v_1^{(1)} v_2^{(2)} v_2^{(1)} \dots v_{(k-1)/2}^{(2)} v_{(k-1)/2}^{(1)} v_{(k+1)/2}^{(2)}$. Then $F = P_k$ is a P_k -factor. Increasing all point numbers of F in V_2 by 2 (mod n) simultaneously $n/2$ times, we obtain $n/2$ P_k -factors. Then it can be easily checked that these P_k -factors are line-disjoint and that the sum of them is a P_k -factorization of $K_{m,n}$.

Lemma 5. $k \equiv 1 \pmod{4}$, $m = k-1$, $n = k+1$

$\Rightarrow K_{m,n}$ has a P_k -factorization.

Proof. The proof is shown by a simple construction algorithm.

Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}\}$, where $m = k-1$ and $n = k+1$. Construct two P_k 's such as $P_k^{(1)} = v_1^{(2)} v_1^{(1)} v_2^{(2)} v_2^{(1)} \dots v_{(k-1)/2}^{(2)} v_{(k-1)/2}^{(1)} v_{(k+1)/2}^{(2)}$ and $P_k^{(2)} = v_{a+1}^{(2)} v_{b+1}^{(1)} v_{a+2}^{(2)} v_{b+2}^{(1)} \dots v_{a+(k-1)/2}^{(2)} v_{b+(k-1)/2}^{(1)} v_{a+(k+1)/2}^{(2)}$, where $a = (k+1)/2$ and $b = (k-1)/2$. Then $F = P_k^{(1)} \cup P_k^{(2)}$ is a P_k -factor. Increasing all point numbers of F in V_2 by 2 (mod n) simultaneously $n/2$ times, we obtain $n/2$ P_k -factors. Then it can be easily checked that these P_k -factors are line-disjoint and that the sum of them is a P_k -factorization of $K_{m,n}$.

Applying Theorem 3 to Lemma 4 - Lemma 5 and considering Lemma 2 - Lemma 3, we have the following Theorems.

Theorem 5. k is odd, $(k-1)m = (k+1)n$ and $k m n / ((k-1)(m+n))$ is an integer

$\Rightarrow K_{m,n}$ has a P_k -factorization.

Theorem 6. k is odd, $(k-1)n = (k+1)m$ and $k m n / ((k-1)(m+n))$ is

an integer

$\implies K_{m,n}$ has a P_k -factorization.

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